Nested utility function exercise

Consider the following utility function:

$$u = x_1 \min\{x_2, x_3\}$$

Subject to the following constraint:

$$M = p_1 x_1 + p_2 x_2 + p_3 x_3$$

1. Find the Marshallian demands. Find the indirect utility function.
2. Find the Hicksian demands and the expenditure function.

Solutions

1. First, it is important to note that at the optimum it can never be the case that $x_2 \neq x_3$ as this would imply spending resources on x_2 or x_3 in excess without any increase in utility. Therefore, we can state that $x_2 = x_3$ and hence the utility function would be:

$$u = x_1 \min\{x_2, x_2\} = x_1 x_2$$

Taking this into account in the constraint:

$$M = p_1 x_1 + p_2 x_2 + p_3 x_2$$

$$M = p_1 x_1 + x_2(p_2 + p_3)$$

And solving the maximization as if it were a conventional Cobb-Douglas, where we can introduce a new price for good 2 as if it were the sum of p_2 and p_3 and once solved replace with said term:

$$M = p_1 x_1 + p x_2$$

The Marshallian demands would then be:

$$x_1^m = \frac{M}{p_1}\frac{1}{2}$$

$$x_2^m = \frac{M}{p_2 + p_3}\frac{1}{2}$$

And as $x_2 = x_3$ then:

$$x_3^m = \frac{M}{p_2 + p_3}\frac{1}{2}$$

The indirect utility function would therefore be:

$$v = \frac{M}{p_1}\frac{1}{2}\min\{\frac{M}{p_2 + p_3}\frac{1}{2}, \frac{M}{p_2 + p_3}\frac{1}{2}\} = \frac{M}{p_1}\frac{1}{2}\frac{M}{p_2 + p_3}\frac{1}{2} = \frac{M^2}{4}\frac{1}{p_1(p_2 + p_3)}$$

2. For the expenditure minimization problem, we can use the same reasoning as before. We know that at the optimum $x_2 = x_3$ and therefore the Hicksian demands are those equal to the conventional Cobb-Douglas except that what is considered the price of x_2 is the sum of prices $p_2 + p_3$:

$$x_1^h = \sqrt{\frac{\bar{u}(p_2 + p_3)}{p_1}}$$

$$x_2^h = x_2^h = \sqrt{\frac{\bar{u}(p_1)}{p_2 + p_3}}$$

And the expenditure function:

$$e = 2\sqrt{p_1(p_2 + p_3)\bar{u}}$$